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# Gaussian Approximation Based Mixture Reduction for Near Optimum Detection in MIMO Systems

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**Abstract**—The optimal “soft” symbol detection for spatial multiplexing multiple input multiple output (MIMO) system with known channel information requires knowledge of the marginal posterior symbol probabilities for each antenna. The calculation of these quantities requires the evaluation of the likelihood function of the system for all possible symbol combinations, which is prohibitive for large systems. It is however most often the case that most of the transmitted symbol combinations contribute only very little to these marginal posterior probabilities. We propose in this paper a suboptimal procedure which identifies the most significant symbol combinations via a sequential algorithm with Gaussian Approximation (SGA). Simulation results show that our method can approach the optimal a posteriori probability detector (APP) performance while being less complex than comparable suboptimal algorithms, such as the sphere decoder (SD). We further demonstrate that as opposed to the SD the complexity and memory requirements of our algorithm are fixed, therefore easing practical implementation.

**Index Terms**—Space-time processing, multiuser detection, Gaussian approximation, probability data association.

## I. INTRODUCTION

THE BENEFITS of MIMO transmission are well appreciated by now. Likewise, a plethora of detection algorithms have been proposed for so-called spatial multiplexing (V-BLAST) schemes. Amongst the frontrunners are various flavours of sphere decoders (SD) [1] and algorithms based on the Probabilistic Data Association (PDA) principle [2] [3] [4] [5]. PDA based algorithms perform well (close to the optimal APP decoder) for simple modulation schemes i.e. BPSK and QPSK, but these results do not carry on to higher order modulations [4]. SD type algorithms tend to perform very well but suffer from the fact that their complexity is a random variable which depends on the channel realization. In this contribution we develop a novel algorithm that builds on the PDA principle, which possesses the advantage of performing near optimally for larger modulations, while having a fixed complexity, typically lower than that of a SD.

## II. SYSTEM MODEL

Consider a spatial multiplexing MIMO system with  $N_T$  transmit antennas and  $N_R \geq N_T$  receive antennas. At each

time instant,  $N_T$  symbols  $\mathbf{x} \stackrel{\text{def}}{=} [x_1, x_2, \dots, x_{N_T}]^T$  ( $[\ast]^T$  means transpose), taken from a modulation constellation  $A = \{a_1, a_2, \dots, a_N\}$ , are transmitted from each antenna. Pertaining to them are  $N_R$  observations  $\mathbf{y} \stackrel{\text{def}}{=} [y_1, y_2, \dots, y_{N_R}]^T$ . The relationship between  $\mathbf{x}$  and  $\mathbf{y}$  is :

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (1)$$

where  $\mathbf{H}$  is the  $N_R \times N_T$  channel matrix with  $h(i, j)$  as its  $(i, j)$ -th entry. The quantity  $h(i, j)$  represents the channel gain from transmit antenna  $j$  to receive antenna  $i$ . Vector  $\mathbf{n}$  is a  $N_R \times 1$  zero-mean complex circular symmetric Gaussian noise with covariance matrix  $\sigma^2 \mathbf{I}$ . We will use  $[\ast]^H$  for the transpose conjugate of a matrix or vector.

The task of a space-time decoder is to estimate the transmitted symbol  $\mathbf{x}$  from the observation  $\mathbf{y}$  given the channel state information  $\mathbf{H}$ . More precisely we are interested in the marginal posterior distributions  $p(x_j|\mathbf{y}, \mathbf{H})$  for  $j = 1, 2, \dots, N_T$  (in what follows, conditioning on  $\mathbf{H}$  will be implicit, and omitted).

The exact computation of the marginal posterior distributions  $p(x_j|\mathbf{y})$  requires an exhaustive search in the space of all possible symbol combinations,

$$p(x_j|\mathbf{y}) = \sum_{\mathbf{x}_{-j} \in D_{-j}} p(\mathbf{x}_{-j}, x_j|\mathbf{y}), \quad (2)$$

where  $\mathbf{x}_{-j}$  refers to all the antennas except antenna  $j$  and  $D_{-j}$  is the set which contains the  $N^{N_T-1}$  possible values of  $\mathbf{x}_{-j}$ . This is often referred to as the a posteriori probability (APP) probability, and requires prohibitive computations for large systems.

However, most of the terms in the sum above are typically very small and contribute very little to the final result. It is therefore natural to look for a subset of  $M$  dominant symbol combinations  $\Theta_{N_T} \stackrel{\text{def}}{=} \{(x_1^{(m)}, \dots, x_{N_T}^{(m)}), m = 1, \dots, M\}$  which will allow for the following truncated sum,

$$\sum_{m=1}^M p(x_1^{(m)}, \dots, x_j, \dots, x_{N_T}^{(m)}|\mathbf{y}), \quad (3)$$

to be a good approximation of  $p(x_j|\mathbf{y})$ .

Note that the selection of these  $M$  most significant symbol combinations would in principle require the computation of the joint posterior distributions of the  $N^{N_T}$  possible symbol combinations, which is precisely what we would like to avoid. To circumvent the complexity problem, we develop a suboptimal mixture reduction method which proceeds in a sequential manner and approximates some of the required quantities using Gaussian approximations.

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### III. IDENTIFICATION OF THE $M$ MOST SIGNIFICANT SYMBOL COMBINATIONS

Assume that for  $j \geq 1$ , at the  $(j-1)$ -th step of the algorithm we have identified  $M$  significant combinations  $\Theta_{j-1} \stackrel{\text{def}}{=} \{x_1^{(m)}, \dots, x_{j-1}^{(m)}, m = 1, 2, \dots, M\}$  for antenna  $1, 2, \dots, j-1$ . We would like to calculate  $p(x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j | \mathbf{y})$  for all  $m = 1, \dots, M$  and  $x_j \in A$  in order to select  $\Theta_j$  which contains  $M$  symbol combinations of the largest probabilities, among the  $MN$  possibilities. However this quantity requires prohibitive computations, and instead we choose a Gaussian approximation.

Provided that  $\mathbf{H}^H \mathbf{H}$  is invertible one can rewrite Eq. (1) as follows,

$$\tilde{\mathbf{y}} = \mathbf{x} + \tilde{\mathbf{n}} = \sum_{k=1}^j x_k \mathbf{e}_k + \sum_{k=j+1}^{N_T} x_k \mathbf{e}_k + \tilde{\mathbf{n}} \stackrel{\text{def}}{=} \sum_{k=1}^j x_k \mathbf{e}_k + \hat{\mathbf{n}}_j, \quad (4)$$

where  $\tilde{\mathbf{n}}$  is a Gaussian noise with zero mean and covariance  $\mathbf{\Lambda} = \sigma^2 (\mathbf{H}^H \mathbf{H})^{-1}$ ,  $\tilde{\mathbf{y}} = (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{y}$ .

Now one models the distribution of  $\hat{\mathbf{n}}_j$  as a Gaussian noise with matching mean and variance. One can then calculate an approximated joint symbol probability  $\tilde{p}(x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j | \mathbf{y})$  of all the  $MN$  possible symbol combinations for  $m = 1, 2, \dots, M$  and  $x_j \in A$

$$\begin{aligned} & \tilde{p}(x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j | \mathbf{y}) \\ & \propto \tilde{p}(\mathbf{y} | x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j) p(x_j) \prod_{k=1}^{j-1} p(x_k^{(m)}) \\ & \approx \exp(-\mathbf{w}^H \mathbf{\Pi}_j^{-1} \mathbf{w}) p(x_j) \prod_{k=1}^{j-1} p(x_k^{(m)}) \stackrel{\text{def}}{=} \psi_m(x_j) \end{aligned} \quad (5)$$

where  $\mathbf{w} = \tilde{\mathbf{y}} - [x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j, x_0, \dots, x_0]^T$ ,  $\mathbf{\Pi}_j = \mathbf{\Lambda} + \gamma \sum_{k=j+1}^{N_T} \mathbf{e}_k \mathbf{e}_k^T$  ( $\mathbf{e}_k$  is a column vector whose elements are all zeroes, but the  $k$ -th which is 1),  $x_0$  and  $\gamma$  are the mean and variance of the modulation alphabet  $A$  with respect to uniform distribution respectively,  $p(x_j)$  is the prior information.

Then  $M$  symbol combinations with the largest  $\psi_m(x_j)$  are selected among the  $MN$  possible symbol combinations, resulting in a new set  $\Theta_j$ .

We illustrate the algorithm with an example shown in Fig. 1. Here  $N_T = 3, N_R \geq 3$  and  $A = \{a_1, a_2, a_3, a_4\}$ . Our aim is to identify the  $M = 2$  most significant symbol combinations from the 64 possibilities. The possible symbol combinations can be represented as a trellis in Fig. 1(a): the two thick lines indicate the actual 2 most significant symbol combinations, *i.e.*  $(a_4, a_4, a_4)$  and  $(a_1, a_3, a_1)$ .

In the first step,  $j = 1$  and we calculate  $\psi_0(x_1)$  for  $x_1 \in A$  according to Eq. (5), *i.e.*  $x_0$  is assumed to have been transmitted on antenna 2 and 3. Assume that  $\psi_0(a_1)$  and  $\psi_0(a_4)$  are the two largest  $\psi_0(x_1)$  for  $x_1 \in A$ : we set  $\Theta_1 = \{a_1, a_4\}$ . This step is illustrated in Fig. 1(b). In the second step  $j = 2$  and we compute the 8 possible values of  $\psi_m(x_2)$  for  $m = 1, 2$  and  $x_2 \in A$  while  $x_0$  is assumed transmitted by antenna 3. Assuming that  $\psi_1(a_3)$  and  $\psi_2(a_4)$  are the 2 largest values among the 8 possibilities, we set  $\Theta_2 = \{(a_4, a_4), (a_1, a_3)\}$ . This step is illustrated in Fig.

1(c). This procedure is again repeated in order to identify  $\Theta_3 = \{(a_4, a_4, a_4), (a_1, a_3, a_1)\}$ .

### IV. ALGORITHM SUMMARY

To sum up, the algorithm proceeds as follows:

- 1) Initialization: Compute  $\tilde{\mathbf{y}}$ ,  $x_0$  and  $\gamma$ , and set  $\Theta_0 = \emptyset$ ,
- 2)  $M$  most significant symbol combinations selection. For  $j = 1$  compute  $\psi_0(x_1)$  for  $x_1 \in A$ .
  - a) For  $1 < j \leq N_T$ , compute  $\psi_m(x_j)$  for all the elements in  $\Theta_{j-1}$  and  $x_j \in A$  according to Eq. (5).
  - b) Select the  $\min(M, N^j)$  symbol combinations which have the largest  $\psi_m(x_j)$  and form the set  $\Theta_j$ .
- 3) Computation of the marginal symbol probability for antenna  $j = 1, 2, \dots, N_T$ :

- a) For  $m = 1, \dots, M$  and  $x_j \in A$  compute

$$\begin{aligned} \phi_m(x_j) &= \exp(-(\mathbf{H}\mathbf{v})^H \mathbf{H}\mathbf{v} / \sigma^2) p(x_j) \prod_{k \neq j} p(x_k^{(m)}), \\ & \text{with } \mathbf{v} = \tilde{\mathbf{y}} - [x_1^{(m)}, \dots, x_{j-1}^{(m)}, x_j, x_{j+1}^{(m)}, \dots, x_{N_T}^{(m)}]^T \end{aligned} \quad (6)$$

- b) Compute the symbol probabilities for  $x_j \in A$ ,

$$\tilde{p}(x_j | \mathbf{y}) = \sum_m \phi_m(x_j) / \sum_{x_j} \sum_m \phi_m(x_j). \quad (7)$$

- c) (Optional) Perform Step 2b with  $\psi_m$  replaced with  $\phi_m$ .

Note that the optional Step 3c was only found useful for small values of  $M$ , and did not lead to improvements otherwise. It is typically beneficial in circumstances where significant paths might have been prematurely deleted.

### V. COMPLEXITY REDUCTION

The complexity of SGA can be reduced to  $O(N_T^3)$  by using the Sherman-Morrison-Woodbury formula in order to sequentially update  $\mathbf{\Pi}_j^{-1}$  in Eq. (5). More precisely, for  $j = 1, \dots, N_T - 2$ , we define respectively  $\mathbf{p}_j$  and  $\pi_{(j,j)}$  the  $j$ -th column and  $j$ -th diagonal element of  $\mathbf{\Pi}_j^{-1}$ . The backward recursion proceeds as follows

$$\mathbf{\Pi}_j^{-1} = (\mathbf{\Pi}_{j+1} + \gamma \mathbf{e}_{j+1} \mathbf{e}_{j+1}^T)^{-1} = \mathbf{\Pi}_{j+1}^{-1} - \frac{\gamma \mathbf{p}_{j+1} \mathbf{p}_{j+1}^H}{1 + \gamma \pi_{(j+1,j+1)}}, \quad (8)$$

The recursion is initialized with  $\mathbf{\Pi}_{N_T-1}^{-1}$  given by:

$$\mathbf{\Pi}_{N_T-1}^{-1} = (\mathbf{\Lambda} + \gamma \mathbf{e}_{N_T} \mathbf{e}_{N_T}^T)^{-1} = \frac{1}{\sigma^2} \mathbf{H}^H \mathbf{H} - \frac{\gamma \varpi \varpi^H}{\sigma^4 + \gamma \sigma^2 \zeta}, \quad (9)$$

where  $\varpi$  is the  $N_T$ -th column of  $\mathbf{H}^H \mathbf{H}$  and  $\zeta$  is the  $N_T$ -th diagonal element of  $\mathbf{H}^H \mathbf{H}$ . Note also that the calculations of Steps 2a and 3a, which are the most time consuming, are suitable for parallel implementation.

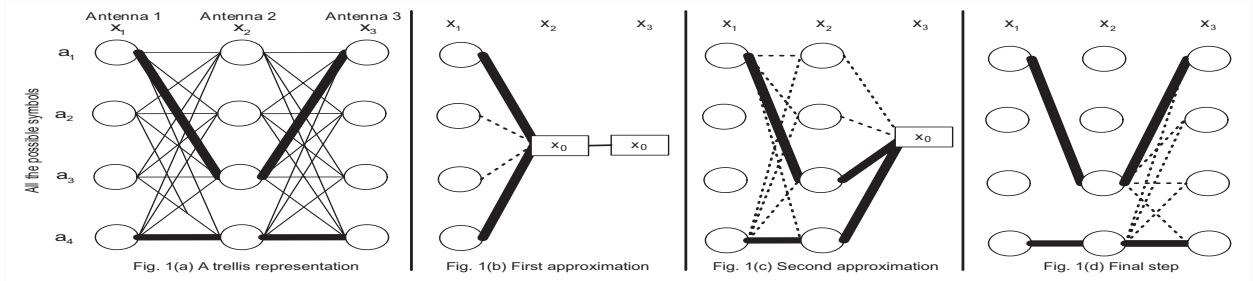


Fig. 1. An example of identification of the 2 most significant symbol combinations via SGA.

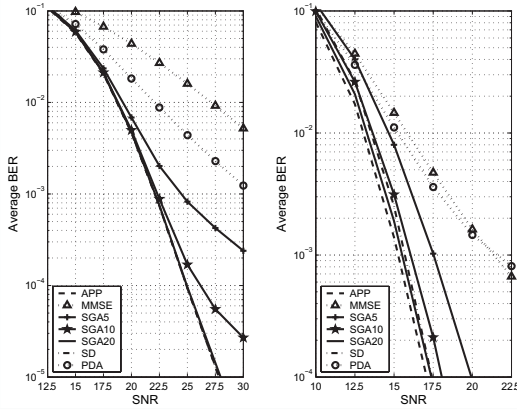


Fig. 2. Left: uncoded system. Right: coded system.

## VI. SIMULATION RESULTS

In this section, we compare the performance of SGA for different values of  $M$  with that of the optimal APP decoder, conventional MMSE and the PDA detector. We also compare the complexity of SGA with that of SD.

In all our simulations, we set  $N_T = N_R = 4$  and consider a 16QAM modulation ( $N = 16$ ) with 1152 bits per frame before channel coding. The SNR is defined as  $E\{\|\mathbf{H}\mathbf{x}\|^2\}/E\{\|\mathbf{n}\|^2\} = \gamma N_T/\sigma^2$ .

A 1/2 rate Turbo Coder with generators 7 and 5 in octal notation is used at the transmitter and a BCJR channel decoder with 4 iterations is used at the receiver. There are no outer iterations, i.e. the MIMO decoder processes the data only once. For each SNR we randomly generated  $10^4$  channel realizations, which were processed by all algorithms. The uncoded and coded performance of the APP, MMSE, PDA and SGA algorithms with  $M \in \{5, 10, 20\}$  is presented in Fig. 2. It can be seen that the performance of SGA with  $M = 20$  approaches that of APP in both the coded and uncoded cases.

Similar near optimal performance can be obtained with the SD [1], but in contrast to the SGA, its complexity is affected by the channel realizations. In order to illustrate this we have carried out a comparative Monte Carlo study of the distribution of the number of real operations (ADD+MUL) of SGA and a Max-Log-MAP efficient implementation of SD [6] using the setting described above. The results are summarized in Fig. 3 where we present the 20, 40, 60 and 80 percentiles of (ADD+MUL) for SD and the constant (ADD+MUL) for SGA for different values of  $M$ , both as a function of the SNR. We observe that the complexity of SD is much larger than that of SGA, especially at low SNR levels.

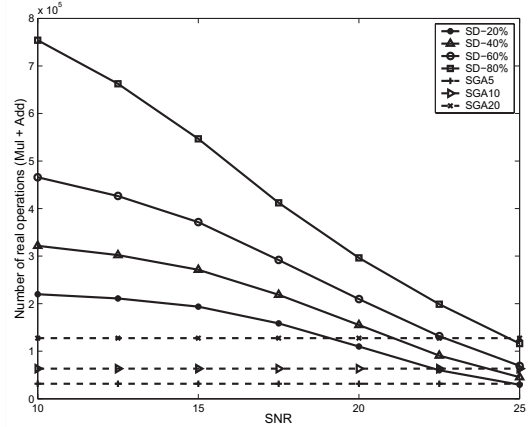


Fig. 3. Complexity comparison.

## VII. CONCLUSIONS

We present a new sub-optimal algorithm for space time decoding of MIMO systems, based on the identification of  $M$  most significant symbol combinations and a likelihood approximation. Simulation results demonstrate that our algorithm achieves near optimal performance with the advantage of having complexity which is both a constant as a function of SNR and lower than that of the popular SD algorithm, especially at low SNR levels. Finally the structure of the algorithm makes it ideal for parallel implementation.

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